Regularly log-periodic functions and some applications

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Abstract

We prove a Tauberian theorem for the Laplace–Stieltjes transform and Karamata-type theorems in the framework of regularly log-periodic functions. As an application we determine the exact tail behavior of fixed points of certain type smoothing transforms.
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1 Introduction

A function $f : [0, \infty) \to [0, \infty)$ is *regularly log-periodic*, $f \in \mathcal{RL}$ or $f \in \mathcal{RL}(p, r, \rho)$, if it is measurable, there is a slowly varying function at infinity ℓ, real numbers $\rho \in \mathbb{R}$, $r > 1$, and a positive logarithmically periodic function $p \in \mathcal{P}_r$, such that

$$\lim_{n\to\infty} \frac{f(xr^n)}{(xr^n)^\rho \ell(xr^n)} = p(x), \quad x \in C_p, \tag{1}$$

where C_p stands for the set of continuity points of p, and for $r > 1$

$$\mathcal{P}_r = \Big\{ p : (0, \infty) \to (0, \infty) : \inf_{x\in[1,r]} p(x) > 0, \ p \text{ is bounded, right-continuous,} \\ \text{and } p(xr) = p(x), \ \forall x > 0 \Big\}.$$

This function class is a natural and important extension of regularly varying functions, and it appears in different areas of theoretical and applied probability. This class arises in connection with various random fixed point equations, such as the smoothing transformation. Regularly log-periodic functions are the basic ingredients in the theory of semistable laws. The tail of the limiting random variable of a supercritical Galton–Watson process is also regularly log-periodic. These are spelled out in details in Section 3. Here we only mention some results for the perpetuity equation

$$X \overset{\mathcal{D}}{=} AX + B, \tag{2}$$

where (A, B) and X on the right-hand side are independent. Under appropriate assumptions, Grincevičius [17, Theorem 2] showed that the tail of the solution of (2) is regularly log-periodic with constant slowly varying function. Under similar assumptions the same asymptotic behavior was shown for the max-equation $X \stackrel{\mathcal{D}}{=} \max\{AX, B\}$, which corresponds to the maximum of perturbed random walks; see Iksanov [22, Theorem 1.3.8]. More generally, this type of tail behavior appears in implicit renewal theory in the arithmetic case; see Jelenković and Olvera-Cravioto [23, Theorem 3.7], and Kevei [24]. In general, functions of the form $p(x)e^{\lambda x}$, $\lambda \in \mathbb{R}$, where p is a periodic function, are solutions of certain integrated Cauchy functional equations, see Lau and Rao [26]. For physical relevance of log-periodicity we refer to Sornette [32].

The name 'regularly log-periodic' comes from Buldygin and Pavlenkov [9, 10], where a function f is called regularly log-periodic, if

$$f(x) = x^\rho \ell(x) p(x), \quad x > 0, \tag{3}$$

where ℓ, ρ and r are the same as above, and $p \in \mathcal{P}_r$ is *continuous*. This condition is clearly much stronger than (1) even without the continuity of p. In the examples given above, the continuity assumption does not necessarily hold, and this is the reason for the extension of the definition. Moreover, our main motivation originates in the studies of the St. Petersburg distribution, where the corresponding p function is not continuous; see Example 2 at the end of Subsection 3.1.

In what follows, we assume that $U : [0, \infty) \to [0, \infty)$ is a nondecreasing function, and

$$\widehat{U}(s) = \int_0^\infty e^{-sx} \mathrm{d}U(x)$$

denotes its Laplace–Stieltjes transform. Since we need monotonicity, for $r > 1$ we further introduce the sets of functions

$$\begin{aligned} \mathcal{P}_{r,\rho} &= \Big\{ p : (0,\infty) \to (0,\infty) \, : \, p \in \mathcal{P}_r, \text{ and } x^\rho p(x) \text{ is nondecreasing} \Big\}, \ \rho \geq 0, \\ \mathcal{P}_{r,\rho} &= \Big\{ p : (0,\infty) \to (0,\infty) \, : \, p \in \mathcal{P}_r, \text{ and } x^\rho p(x) \text{ is nonincreasing} \Big\}, \ \rho < 0. \end{aligned} \tag{4}$$

In order to characterize the Laplace–Stieltjes transform of regularly log-periodic functions, for $r > 1$, $\rho \geq 0$, put

$$\mathcal{Q}_{r,\rho} = \Big\{ q : (0,\infty) \to (0,\infty) \, : \, s^{-\rho} q(s) \text{ is completely monotone, and } q(sr) = q(s), \ \forall s > 0 \Big\}. \tag{5}$$

For $\rho = 0$ the sets $\mathcal{P}_{r,0}, \mathcal{Q}_{r,0}$ are just the set of constant functions.

The aim of the present paper is to prove Tauberian theorem for the Laplace–Stieltjes transform, and Karamata-type theorems in the framework of regularly log-periodic functions. The ratio Tauberian theorem [8, Theorem 2.10.1], a general version of the Tauberian theorem for Laplace-Stieltjes transforms, holds for O-regular varying functions. The equivalence of the behavior of U at infinity and $\widehat{U}$ at zero holds, if and only if $U^*(\lambda) = \limsup_{x\to\infty} U(\lambda x)/U(x)$ is continuous at 1. The latter condition for functions defined in (3) is equivalent to the continuity of p; see Proposition 2. In particular, the discontinuity of p is the reason that the ratio Tauberian theorem [8, Theorem 2.10.1] does not hold in this setup. However, in Theorem 1 below we do provide an equivalence between the tail behavior of the function, and the behavior of its Laplace–Stieltjes transform at zero. In [9, 10], Buldygin and Pavlenkov proved Karamata theorems in the sense of Theorems 1.5.11 (direct half) and 1.6.1 (converse half) of Bingham, Goldie and Teugels [8], for functions satisfying (3) with continuous p. Here we extend these results.

Section 2 contains the main results of the paper. After some preliminaries, first we deal with a Tauberian theorem for the Laplace–Stieltjes transform, then we prove the direct half of the Karamata theorem, and a monotone density theorem. In Section 3 we give some applications. We prove that the tail of a nonnegative random variable is regularly log-periodic, if and only if the same is true for its Laplace transform at 0. Using this result we determine the tail behavior of fixed points of certain smoothing transforms. We reprove, in a special case, a result by Watanabe and Yamamuro [35] for tails of semistable random variables. Finally, we spell out some related results on the limit of supercritical branching processes.

2 Results

2.1 Preliminaries

First we discuss the place of regularly log-periodic functions among well-known function classes, such as regularly varying functions, extended and O-regularly varying functions.

In the following we always assume that $f : [0, \infty) \to [0, \infty)$ is nonnegative and measurable. For $\lambda > 0$ let

$$f^*(\lambda) = \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \quad f_*(\lambda) = \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)}.$$

A function f is *extended regularly varying* if for some constants c, d

$$\lambda^d \le f_*(\lambda) \le f^*(\lambda) \le \lambda^c, \ \lambda > 1, \tag{6}$$

and it is *O-regularly varying* if

$$0 < f_*(\lambda) \le f^*(\lambda) < \infty.$$

First we note that general regularly log-periodic functions can be quite irregular.

Example 1. Consider the function

$$f(x) = \begin{cases} n, & \text{if } x \in [(1 + n^{-1})2^n, (1 + 2n^{-1})2^n], \ n \ge 2, \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

Then (1) holds with $\ell(x) \equiv 1$, $\rho = 0$, $r = 2$, and $p(x) \equiv 1$. Indeed, $\lim_{n\to\infty} f(2^n x) = 1$ for every $x > 0$, but f is not even bounded, and the exceptional intervals are large.

For monotone log-periodic functions the situation is not so bad. A function $f : [0, \infty) \to [0, \infty)$ is ultimately monotone if it is monotone (increasing or decreasing) for large enough x.

Proposition 1. *Let $f \in \mathcal{RL}(p, r, \rho)$ be an ultimately monotone regularly log-periodic function. Then*

$$\limsup_{x\to\infty} \frac{f(x)}{x^\rho \ell(x)} < \infty,$$

and f is O-regularly varying.

Proof. Assume that f is ultimately monotone increasing. The decreasing case follows the same way. Indirectly assume that $f(x_n)/(x_n^\rho \ell(x_n)) \to \infty$ for some $x_n \uparrow \infty$. Write $x_n = r^{k_n} z_n$, where $z_n \in [1, r)$. Using the Bolzano–Weierstrass theorem, we may assume that $z_n \to \lambda \in [1, r]$. With some $\lambda < \eta \in C_p$, for large enough n

$$\frac{f(r^{k_n} z_n)}{(r^{k_n} z_n)^\rho \ell(r^{k_n} z_n)} \leq \frac{f(r^{k_n} \eta)}{(r^{k_n})^\rho \ell(r^{k_n} z_n)} \to \eta^\rho p(\eta),$$

which is a contradiction. The O-regular variation follows from the boundedness and strict positivity of p. $\square$

For the extended regular variation, and for the continuity of f^* stronger conditions are needed.

Proposition 2. *Assume that for a slowly varying function ℓ, for $\rho \in \mathbb{R}$, $r > 1$, and $p \in \mathcal{P}_r$*

$$f(x) = x^\rho \ell(x) p(x).$$

Then f is

(i) extended regularly varying if and only if p is Lipschitz on $[1, r]$;

(ii) regularly varying if and only if p is constant.

Moreover, f^* is continuous at 1, if and only if p is continuous.

Note that a logarithmically periodic function is globally Lipschitz if and only if it is constant.

Proof of Proposition 2. The logarithmic periodicity of p implies

$$f^*(\lambda) = \lambda^\rho \sup_{x \in [1,r]} \frac{p(\lambda x)}{p(x)},$$

from which we see that f^* is continuous at 1 if and only if p is continuous.

We turn to (i). Let $\lambda > 1$. If p is Lipschitz with Lipschitz constant L, then for $x \in [1, r]$ we have $p(\lambda x) \leq p(x) + Lx(\lambda - 1)$, thus

$$\sup_{x \in [1,r]} \frac{p(\lambda x)}{p(x)} \leq 1 + L(\lambda - 1) \sup_{x \in [1,r]} \frac{x}{p(x)} \leq \lambda^{c-\rho}$$

for some $c > 0$. The proof of the lower bound is similar. For the converse, assume indirectly that p is not Lipschitz. Then there are two sequences $\lambda_n \downarrow 1$, and $x_n \to x \in [1, r]$ such that

$$|p(\lambda_n x_n) - p(x_n)| \geq n x_n (\lambda_n - 1),$$

consequently (6) cannot hold. Finally, (ii) is obvious. $\square$

2.2 Tauberian theorem for the Laplace transform

Recall (4) and (5). There is a natural correspondence between $\mathcal{P}_{r,\rho}$ and $\mathcal{Q}_{r,\rho}$.

Lemma 1. *For $p \in \mathcal{P}_{r,\rho}$, $\rho > 0$, define the operator $\mathrm{A}_{r,\rho} = \mathrm{A}_\rho$ as*

$$\mathrm{A}_\rho p(s) = s^\rho \int_0^\infty e^{-sx} \mathrm{d}(p(x)x^\rho). \tag{8}$$

Then $\mathrm{A}_\rho : \mathcal{P}_{r,\rho} \to \mathcal{Q}_{r,\rho}$ is one-to-one.

Proof of Lemma 1. It is clear from the definition that $\mathrm{A}_\rho p \in \mathcal{Q}_{r,\rho}$.

Conversely, let $q \in \mathcal{Q}_{r,\rho}$ be given. Since $s^{-\rho}q(s)$ is completely monotone, there is a nondecreasing right-continuous function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$s^{-\rho}q(s) = \int_0^\infty e^{-sx} \mathrm{d}g(x). \tag{9}$$

To prove that $p(x) := x^{-\rho}g(x) \in \mathcal{P}_{r,\rho}$ we only have to show the logarithmic periodicity of p. Substituting $s \to rs$ in (9) and using that $q(rs) = q(s)$ we obtain that

$$\int_0^\infty e^{-sx} \mathrm{d}g(x) = \int_0^\infty e^{-sx} \mathrm{d}[r^\rho g(x/r)].$$

Uniqueness of the Laplace–Stieltjes transform implies

$$g(x) = r^\rho g(x/r), \quad x \in C_g,$$

from which

$$p(x) = p(x/r), \quad x \in C_p.$$

If two right-continuous functions agree in all but countable many points, then they agree everywhere. $\square$

For a real function f the set of its continuity points is denoted by C_f. In the following, ℓ stands for a slowly varying function either at infinity, or at zero. The set of slowly varying functions at infinity (zero) is denoted by $\mathcal{SV}_\infty$ ($\mathcal{SV}_0$).

Theorem 1. *Let $U : [0,\infty) \to [0,\infty)$ be an increasing function, $\rho \geq 0$, $r > 1$, and $\ell \in \mathcal{SV}_\infty$ be a slowly varying function. Then*

$$\lim_{n\to\infty} \frac{U(r^n z)}{(r^n z)^\rho \ell(r^n z)} = p(z) \quad \text{for each } z \in C_p, \text{ for some } p \in \mathcal{P}_r, \tag{10}$$

and

$$\widehat{U}(s) \sim s^{-\rho}\ell(1/s)q(s) \quad \text{as } s \downarrow 0, \text{ for some } q \in \mathcal{P}_r, \tag{11}$$

are equivalent. In each case, necessarily $p \in \mathcal{P}_{r,\rho}$, $q \in \mathcal{Q}_{r,\rho}$, and $\mathrm{A}_\rho p = q$ for $\rho > 0$, and $p = q$ for $\rho = 0$.

Moreover, if p is continuous, then (10) implies

$$U(x) \sim x^\rho \ell(x)p(x) \quad \text{as } x \to \infty. \tag{12}$$

Remark 1. (i) For $\rho = 0$ the result follows from [8, Theorem 1.7.1].

(ii) The equivalence of $U(r^n z) = o(r^n \ell(r^n))$ and $\widehat{U}(s) = o(s^{-\rho}\ell(1/s))$ also follows from [8, Theorem 1.7.1].

(iii) For continuous p the ratio Tauberian theorem [8, Theorem 2.10.1], (Korenblyum [25], Feller [15], Stadtmüller and Trautner [33]) states that (11) and (12) are equivalent. Indeed, by Propositions 1 and 2 U is always O-regularly varying and p is continuous if and only if $U^*(\lambda)$ is continuous at 1. Moreover, the Laplace–Stieltjes transform of $x^\rho p(x)$ is $s^{-\rho} q(s)$. Theorem 2.10.1 (iii) in [8] states that the continuity of U^* at 1, is also necessary in general for the equivalence of (11) and (12).

Proof of Theorem 1. Concerning the first remark above, we may assume that $\rho > 0$. The proof follows the standard idea of Tauberian theorems (see Theorem 1.7.1 [8]) combined with the following lemma from [24].

Lemma 2. *Assume that $p \in \mathcal{P}_r$ is continuous, $\ell \in \mathcal{SV}_\infty$, $\alpha \in \mathbb{R}$, U is monotone, and for any $z \in [1, r)$*

$$\lim_{n\to\infty} \frac{U(zr^n)}{(zr^n)^\alpha \ell(r^n)} = p(z).$$

Then

$$U(x) \sim x^\alpha \ell(x) p(x).$$

The monotonicity of U and (10) readily imply that $p \in \mathcal{P}_{r,\rho}$. From Proposition 1

$$\limsup_{x\to\infty} \frac{U(x)}{x^\rho \ell(x)} = K < \infty \tag{13}$$

follows. Using Potter's bounds we obtain

$$\begin{aligned}
\widehat{U}(x^{-1}) &= \int_0^\infty e^{-y/x} \mathrm{d}U(y) \\
&\le U(x) + \sum_{n=1}^\infty e^{-2^{n-1}} U(2^n x) \\
&\le 2K x^\rho \ell(x) \left[1 + \sum_{n=1}^\infty e^{-2^{n-1}} 2^{n(\rho+1)} \right].
\end{aligned}$$

Therefore $\widehat{U}(x^{-1})/(x^\rho \ell(x))$ is bounded. Introduce the notation

$$U_x(y) = \frac{U(xy)}{x^\rho \ell(x)}.$$

Using the logarithmic periodicity, for any $z > 0$ we have

$$\lim_{n\to\infty} U_{r^n z}(y) = y^\rho p(zy) =: V_z(y) \quad \text{for all } y \text{ such that } zy \in C_p.$$

Simply

$$\widehat{U}_x(s) = \frac{\widehat{U}(s/x)}{x^\rho \ell(x)}.$$

Since $U_{r^n z}(y)$ converges, using the continuity and uniqueness theorem for Laplace–Stieltjes transforms, we obtain that

$$\lim_{n\to\infty} \frac{\widehat{U}(s/(r^n z))}{(r^n z)^\rho \ell(r^n z)} = \widehat{V}_z(s)$$

for all $s > 0$, since $\widehat{V}_z$, being a Laplace–Stieltjes transform, is continuous. Choosing $s = 1$, after short calculation we have

$$\lim_{n\to\infty} \frac{\widehat{U}(1/(r^n z))}{(r^n z)^\rho \ell(r^n z)} = q(1/z),$$

with $q = \mathrm{A}_\rho p$. Since q is continuous, Lemma 2 implies

$$\widehat{U}(s) \sim s^{-\rho} \ell(1/s) q(s) \quad \text{as } s \downarrow 0,$$

as stated.

For the converse, note that (11) implies

$$\widehat{U}_x(s) = \frac{\widehat{U}(s/x)}{x^\rho \ell(x)} \sim s^{-\rho} q(s/x) \quad \text{as } x \to \infty.$$

Since $q \in \mathcal{P}_r$ we have for any $z > 0$

$$\lim_{n\to\infty} \widehat{U}_{r^n z}(s) = s^{-\rho} q(s/z). \tag{14}$$

Therefore, the continuity theorem gives

$$\lim_{n\to\infty} U_{r^n z}(y) = u_z(y), \quad y \in C_{u_z}$$

for some nondecreasing function u_z. Thus $\widehat{u}_z(s) = s^{-\rho} q(s/z)$, which implies $q \in \mathcal{Q}_{r,\rho}$. Short calculation shows that the right-hand side of (14) is the Laplace–Stieltjes transform of $u_z(y) := y^\rho p(zy)$. Note that $1 \in C_{u_z}$ whenever $z \in C_p$, thus (10) holds for all $z \in C_p$. The second statement follows from Lemma 2. $\square$

The same proof gives analogous result in the case $x \downarrow 0$, $s \to \infty$; see [8, Theorem 1.7.1'].

2.3 Karamata and monotone density theorems

Let $\mathcal{P}_{r,\rho}^m$ denote the set of functions in $\mathcal{P}_{r,\rho}$, which are m-times differentiable on $(0, \infty)$ (we do not assume continuity of the mth derivative). For $r > 1$ and $\rho > 0$ introduce the operator $\mathrm{B}_{r,\rho} = \mathrm{B}_\rho : \mathcal{P}_r \to \mathcal{P}_{r,\rho}^1$

$$\mathrm{B}_\rho p(x) = x^{-\rho} \int_0^x y^{\rho-1} p(y) \mathrm{d}y. \tag{15}$$

Using the logarithmic periodicity, short calculation shows that

$$\int_0^{r^m} s^{\rho-1} p(s) \mathrm{d}s = \frac{r^{m\rho}}{r^\rho - 1} \int_1^r s^{\rho-1} p(s) \mathrm{d}s,$$

and thus

$$\mathrm{B}_\rho p(x) = r^{-\rho\{\log_r x\}} \left[\frac{1}{r^\rho - 1} \int_1^r s^{\rho-1} p(s) \mathrm{d}s + \int_1^{r^{\{\log_r x\}}} s^{\rho-1} p(s) \mathrm{d}s \right], \tag{16}$$

where $\{x\} = x - \lfloor x \rfloor$ stands for the fractional part of x. It is easy to see that $\mathrm{B}_\rho p \in \mathcal{P}^1_{r,\rho}$. Moreover, it is one-to-one with inverse

$$\mathrm{B}_\rho^{-1} q(x) = x^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}x} [x^\rho q(x)], \quad q \in \mathcal{P}^1_{r,\rho}. \tag{17}$$

The following statement is a Karamata type theorem for regularly log-periodic functions; see [8, Theorem 1.5.11].

Theorem 2. *Assume that for some $\rho > 0$,*

$$\lim_{n\to\infty} \frac{u(r^n z)}{(r^n z)^{\rho-1} \ell(r^n z)} = p_0(z) \quad \text{for each } z \in C_{p_0}, \text{ for some } p_0 \in \mathcal{P}_r, \tag{18}$$

and

$$\limsup_{x\to\infty} \frac{u(x)}{x^{\rho-1}\ell(x)} < \infty. \tag{19}$$

Then

$$U(x) = \int_0^x u(y) \mathrm{d}y \sim x^\rho \ell(x) p(x) \quad \text{as } x \to \infty, \tag{20}$$

where $p = \mathrm{B}_\rho p_0$.

Remark 2. (i) For continuous p_0, condition

$$u(x) \sim x^{\rho-1} \ell(x) p_0(x) \quad \text{as } x \to \infty,$$

implies (20); see Lemma 3 by Buldygin and Pavlenkov [10]. (Compare with formula (16). Note that our ρ and their ρ are different.)

(ii) It is again straightforward to extend this result to the case when the limit in (18) is zero.

Proof of Theorem 2. From (19) we readily obtain as in [8, Proposition 1.5.8] that

$$\limsup_{x\to\infty} \frac{U(x)}{x^\rho \ell(x)} < \infty. \tag{21}$$

Short calculation gives for any $0 < \varepsilon < 1$

$$\frac{U(r^n z) - U(r^n z\varepsilon)}{(r^n z)^\rho \ell(r^n z)} = \int_\varepsilon^1 \frac{u(r^n z t)}{(r^n z t)^{\rho-1} \ell(r^n z t)} t^{\rho-1} \frac{\ell(r^n z t)}{\ell(r^n z)} \mathrm{d}t.$$

Whenever $tz \in C_p$ the integrand converges to $p_0(tz)t^{\rho-1}$. Since the set of discontinuity points of a right-continuous function is at most countable, and integrable majorant exists by (19) we see

$$\lim_{n\to\infty} \frac{U(r^n z) - U(r^n z\varepsilon)}{(r^n z)^\rho \ell(r^n z)} = \int_\varepsilon^1 t^{\rho-1} p_0(tz)\,\mathrm{d}t.$$

Finally, (21) implies

$$\limsup_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \frac{U(r^n z\varepsilon)}{(r^n z)^\rho \ell(r^n z)} = 0.$$

Combining the latter two limit relations we obtain

$$\lim_{n\to\infty} \frac{U(r^n z)}{(r^n z)^\rho \ell(r^n z)} = \int_0^1 t^{\rho-1} p_0(tz)\mathrm{d}t = z^{-\rho} \int_0^z s^{\rho-1} p_0(s)\mathrm{d}s = \mathrm{B}_\rho p_0(z). \tag{22}$$

Since $\mathrm{B}_\rho p_0$ is continuous, the statement follows from Lemma 2. □

The statement remains true for $\rho = 0$ in the following version.

Lemma 3. *Assume that for some* $p_0 \in \mathcal{P}_r$

$$\lim_{n\to\infty} \frac{r^n z\, u(r^n z)}{\ell(r^n z)} = p_0(z) \quad \textit{for each } z \in C_{p_0}, \tag{23}$$

and

$$0 < \liminf_{x\to\infty} \frac{xu(x)}{\ell(x)} \le \limsup_{x\to\infty} \frac{xu(x)}{\ell(x)} < \infty. \tag{24}$$

Then $U(x) = \int_0^x u(y)\mathrm{d}y$ *is slowly varying, and* $\lim_{x\to\infty} U(x)/\ell(x) = \infty$.

Remark 3. As for Theorem 2, condition (24) is not very restrictive, and necessary in general.

Proof. The proof is almost identical to the proof of [8, Proposition 1.5.9a].

Put

$$\liminf_{x\to\infty} \frac{xu(x)}{\ell(x)} =: k > 0.$$

Then

$$\liminf_{x\to\infty} \frac{U(x)}{\ell(x)} \ge \frac{k}{2} \liminf_{x\to\infty} \frac{1}{\ell(x)} \int_{\varepsilon x}^x \frac{\ell(y)}{y}\mathrm{d}y = \frac{k}{2} \log \varepsilon^{-1}.$$

As $\varepsilon \downarrow 0$ we get $\lim_{x\to\infty} U(x)/\ell(x) = \infty$. Put $\varepsilon(x) = xu(x)/U(x)$. We showed that $\lim_{x\to\infty} \varepsilon(x) = 0$. Noticing

$$\frac{\mathrm{d}}{\mathrm{d}x} \log U(x) = \frac{U'(x)}{U(x)} = \frac{\varepsilon(x)}{x},$$

the representation theorem of slowly varying functions ([8, Theorem 1.3.1]) implies the statement. □

The converse part of Theorem 2 is the corresponding monotone density theorem.

Theorem 3. *Assume that* $U(x) = \int_0^x u(y)\mathrm{d}y$*, u is ultimately monotone, and (10) holds with* $\rho \geq 0$*. If* $\rho > 0$*, then* $p = \mathrm{B}_\rho p_0$ *for some* $p_0 \in \mathcal{P}_r$*. For* $\rho = 0$ *let* $p_0(x) \equiv 0$*. In both cases*

$$\lim_{n\to\infty} \frac{u(r^n z)}{(r^n z)^{\rho-1}\ell(r^n z)} = p_0(z) \quad \textit{for each } z \in C_{p_0}.$$

Moreover, if p_0 *is continuous, then*

$$u(x) \sim x^{\rho-1}\ell(x)p_0(x) \quad \textit{as } x \to \infty.$$

Remark 4. (i) We see from the statement that if (10) holds, and U has an ultimately monotone density, then necessarily p in (10) is differentiable.

(ii) Note that for $\rho = 0$ the statement follows from the 'usual' monotone density theorem [8, Theorem 1.7.2], since $p \in \mathcal{P}_r$ is necessarily constant. Theorem 1.7.2 in [8] also implies that the result remains true when the limit p in (10) is zero.

Proof of Theorem 3. By (10)

$$\frac{U(bx) - U(ax)}{x^\rho \ell(x)} = \int_a^b \frac{u(sx)}{x^{\rho-1}\ell(x)}\mathrm{d}s$$

is bounded as $x \to \infty$. Since u is ultimately monotone, this readily implies that the integrand is bounded too as $x \to \infty$, which allows us to use Helly's selection theorem. Fix $z > 0$, and consider the sequence $r^n z$. By the selection theorem, there is a subsequence n_k and a monotone limit function v_z such that

$$\lim_{k\to\infty} \frac{u(r^{n_k} zs)}{(r^{n_k} z)^{\rho-1}\ell(r^{n_k} z)} = v_z(s) \quad \text{for each } s \in C_{v_z}. \tag{25}$$

On the other hand, $U(xy)/(x^\rho \ell(x))$ converges on the sequence $r^n z$, thus for the limit function v_z

$$\int_a^b v_z(s)\mathrm{d}s = b^\rho p(bz) - a^\rho p(az) \tag{26}$$

for $0 < a < b < \infty$ such that $az, bz \in C_p$. This clearly determines the limit function in its continuity points, and so the convergence in (25) holds along the whole sequence n. The latter implies that $v_z(rs) = r^{\rho-1}v_z(s)$. From (26) we have that $p \in \mathcal{P}^1_{r,\rho}$. Let $p_0 = \mathrm{B}_\rho^{-1}p$. By (17)

$$v_z(s) = \frac{\mathrm{d}}{\mathrm{d}s}(s^\rho p(sz)) = s^{\rho-1}p_0(sz).$$

If $z \in C_{p_0}$, then $s = 1$ is a continuity point of v_z in (25), and the first statement follows. The second follows from Lemma 2. □

The following statements are versions of the previous results, which we need later. Since the proofs are the same, we omit them.

First we deal with the case when $\rho < 0$. Similarly as before let $\mathcal{P}^1_{r,\rho}$ denote the set of functions in $\mathcal{P}_{r,\rho}$, which are differentiable on $(0,\infty)$. For $r > 1$ and $\rho < 0$ introduce the operator $\mathrm{B}_{r,\rho} = \mathrm{B}_\rho : \mathcal{P}_r \to \mathcal{P}^1_{r,\rho}$

$$\mathrm{B}_\rho p(x) = x^{-\rho} \int_x^\infty y^{\rho-1} p(y) \mathrm{d}y. \tag{27}$$

As before $\mathrm{B}_\rho p \in \mathcal{P}^1_{r,\rho}$, and it is one-to-one with inverse

$$\mathrm{B}_\rho^{-1} q(x) = -x^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}x}[x^\rho q(x)], \quad q \in \mathcal{P}^1_{r,\rho}. \tag{28}$$

Proposition 3. *Let $U(x) = \int_x^\infty u(y)\mathrm{d}y$, where u is ultimately monotone, $r > 1, \rho < 0$. Then*

$$\lim_{n\to\infty} \frac{u(r^n z)}{(r^n z)^{\rho-1}\ell(r^n z)} = p_0(z) \quad \textit{for each } z \in C_{p_0},\ \textit{for some } p_0 \in \mathcal{P}_r,$$

if and only if

$$\lim_{n\to\infty} \frac{U(r^n z)}{(r^n z)^{\rho}\ell(r^n z)} = p(z) \quad \textit{for each } z \in C_p,\ \textit{for some } p \in \mathcal{P}_r.$$

Moreover, $p = \mathrm{B}_\rho p_0$, in particular $p \in \mathcal{P}_{r,\rho}$ is continuous, thus

$$U(x) \sim x^\rho \ell(x) p(x) \quad \textit{as } x \to \infty.$$

For $\rho = 0$ assume further that $\int_0^\infty u(y)\mathrm{d}y < \infty$. Then $U \in \mathcal{SV}_\infty$, and $\lim_{x\to\infty} U(x)/\ell(x) = \infty$.

For continuous p see [10, Lemma 3].

At 0 the corresponding result is the following.

Proposition 4. *Let $U(x) = \int_0^x u(y)\mathrm{d}y$, where u is ultimately monotone, $r > 1, \rho > 0$, and $\ell \in \mathcal{SV}_0$. Then*

$$\lim_{n\to\infty} \frac{u(r^{-n} z)}{(r^{-n} z)^{\rho-1}\ell(r^{-n} z)} = p_0(z) \quad \textit{for each } z \in C_{p_0},\ \textit{for some } p_0 \in \mathcal{P}_r,$$

if and only if

$$\lim_{n\to\infty} \frac{U(r^{-n} z)}{(r^{-n} z)^{\rho}\ell(r^{-n} z)} = p(z) \quad \textit{for each } z \in C_p,\ \textit{for some } p \in \mathcal{P}_r.$$

Moreover, $p = \mathrm{B}_\rho p_0$, in particular p is continuous, thus

$$U(x) \sim x^\rho \ell(x) p(x) \quad \textit{as } x \downarrow 0.$$

3 Applications

3.1 Tails of nonnegative random variables

In this subsection we prove the log-periodic analogue of Theorem A by Bingham and Doney [7] (Theorem 8.1.8 in [8]).

Let X be a nonnegative random variable with distribution function F. If $\mathbf{E}X^m < \infty$, then its Laplace transform

$$\widehat{F}(s) = \int_0^\infty e^{-sx} \mathrm{d}F(x) \tag{29}$$

can be written as

$$\widehat{F}(s) = \sum_{k=0}^{m} \mu_k \frac{(-s)^k}{k!} + o(s^m) \quad \text{as } s \downarrow 0,$$

where $\mu_k = \mathbf{E}X^k$. Define for $m \geq 0$

$$\begin{aligned} f_m(s) &= (-1)^{m+1} \left[\widehat{F}(s) - \sum_{k=0}^{m} \mu_k \frac{(-s)^k}{k!} \right], \\ g_m(s) &= \frac{\mathrm{d}^m}{\mathrm{d}s^m} f_m(s) = \mu_m - (-1)^m \widehat{F}^{(m)}(s). \end{aligned} \tag{30}$$

Theorem 4. *Let $\ell \in \mathcal{SV}_\infty$, $m \in \{0, 1, \ldots\}$, $\alpha = m + \beta$, $\beta \in [0,1]$, $\tilde{q}_m, q_m, p \in \mathcal{P}_r$. The following are equivalent:*

$$f_m(s) \sim s^\alpha \ell(1/s) \tilde{q}_m(s) \text{ as } s \downarrow 0; \tag{31}$$

$$g_m(s) \sim s^\beta \ell(1/s) q_m(s) \text{ as } s \downarrow 0; \tag{32}$$

$$\begin{cases} \lim_{n\to\infty} \ell(r^n)^{-1} \int_{r^n z}^\infty y^m \mathrm{d}F(y) = p(z) \text{ for each } z \in C_p, & \beta = 0, \\ \lim_{n\to\infty} \frac{(r^n z)^\alpha}{\ell(r^n z)} \overline{F}(r^n z) = p(z) \text{ for each } z \in C_p, & \beta \in (0,1), \\ \lim_{n\to\infty} \ell(r^n)^{-1} \int_0^{r^n z} y^{m+1} \mathrm{d}F(y) = p(z) \text{ for each } z \in C_p, & \beta = 1. \end{cases} \tag{33}$$

If $\beta > 0$, then (31)–(33) are further equivalent to

$$(-1)^{m+1} \widehat{F}^{(m+1)}(s) \sim s^{\beta-1} \ell(1/s) q_{m+1}(s) \text{ as } s \downarrow 0, \tag{34}$$

and $q_{m+1} = \mathrm{B}_\beta^{-1} q_m$.

Moreover, the relations between the appearing functions are the following:

$$\begin{aligned} &q_m = \mathrm{B}^{-1}_{\alpha-(m-1)} \mathrm{B}^{-1}_{\alpha-(m-2)} \ldots \mathrm{B}^{-1}_\alpha \widetilde{q}_m, \ \beta \in [0,1], \ q_0 = \widetilde{q}_0, \\ &p_{0,m} = \mathrm{B}^{-1}_{1-\beta} \mathrm{A}^{-1}_{1-\beta} q_m, \ \beta \in (0,1), \\ &p = p_{0,m} - m\mathrm{B}_{-m-\beta} p_{0,m}, \ p_{0,m} = p + m\mathrm{B}_{-\beta} p, \ \beta \in (0,1). \end{aligned}$$

If $\beta \in \{0,1\}$, then necessarily $p(x) \equiv p > 0$, $q_m(s) \equiv q_m > 0$, and $p = q_m$.

Since $p(x)$ is constant for $\beta \in \{0,1\}$, by Lemma 2 (33) is further equivalent to $\int_x^\infty y^m \mathrm{d}F(y) \sim p\ell(x)$, and $\int_0^x y^{m+1} \mathrm{d}F(y) \sim p\ell(x)$ as $x \to \infty$, respectively.

Proof of Theorem 4. We follow the proof of Theorem 8.1.8 in [8].

The equivalence of (31) and (32) follows from iterated application of Proposition 4. (Note that the derivatives of f_m are monotone.) We obtain that $q_m = \mathrm{B}^{-1}_{\alpha-(m-1)} \mathrm{B}^{-1}_{\alpha-(m-2)} \ldots \mathrm{B}^{-1}_\alpha \widetilde{q}_m$. Furthermore, for $\beta > 0$ both (31) and (32) are equivalent to (34), and $q_{m+1} = \mathrm{B}_\beta^{-1} q_m$.

Put

$$U_m(x) = \int_0^x \int_t^\infty y^m \mathrm{d}F(y)\mathrm{d}t,$$

and note that $\widehat{U}_m(s) = g_m(s)/s$. Therefore (32) is equivalent to

$$\widehat{U}_m(s) \sim s^{\beta-1}\ell(1/s)q_m(s). \tag{35}$$

For $\beta \in [0,1]$, using Theorem 1 with $\rho = 1-\beta$ this is equivalent to

$$\lim_{n\to\infty} \frac{U_m(r^n z)}{(r^n z)^{1-\beta}\ell(r^n z)} = p_m(z) \quad z \in C_{p_m}, \tag{36}$$

where $p_m = \mathrm{A}_{1-\beta}^{-1} q_m$, for $\beta \neq 1$, and $p_m = q_m$ for $\beta = 1$.

First assume $\beta \in (0,1)$. By Theorems 2 and 3 with $\rho = 1-\beta$, the latter holds if and only if

$$\lim_{n\to\infty} \frac{u_m(r^n z)}{(r^n z)^{-\beta}\ell(r^n z)} = p_{0,m}(z) \quad z \in C_{p_{0,m}}, \tag{37}$$

where $u_m(x) = \int_x^\infty y^m \mathrm{d}F(y)$, and $\mathrm{B}_{1-\beta} p_{0,m} = p_m$. Note that for $m=0$ this is exactly (33). Partial integration gives

$$u_m(x) = x^m \overline{F}(x) + m \int_x^\infty y^{m-1}\overline{F}(y)\mathrm{d}y. \tag{38}$$

If (33) holds, then by Proposition 3 with $\rho = -\beta$, we obtain (37) with $p_{0,m} = p + m\mathrm{B}_{-\beta} p$, so (32) follows.

Conversely, using Fubini's theorem, we get

$$\frac{x^m \overline{F}(x)}{u_m(x)} = 1 - \frac{m x^m}{u_m(x)} \int_x^\infty y^{-m-1} u_m(y)\mathrm{d}y. \tag{39}$$

Now, Proposition 3 with $\rho = -m-\beta$ shows that (37) is further equivalent to

$$\lim_{n\to\infty} \frac{\int_{r^n z}^\infty y^{-m-1} u_m(y)\mathrm{d}y}{(r^n z)^{-m-\beta}\ell(r^n z)} = \mathrm{B}_{-m-\beta} p_{0,m}(z). \tag{40}$$

Thus, if (37) holds, then by (39)

$$\lim_{n\to\infty} \frac{(r^n z)^{m+\beta}}{\ell(r^n z)} \overline{F}(r^n z) = p_{0,m}(z) - m\mathrm{B}_{-m-\beta} p_{0,m}(z) \quad z \in C_{p_{0,m}},$$

which is exactly (33).

For $\beta = 0$ conditions (36) and (37) are still equivalent. If (37) holds, then the monotonicity of u forces that $p_{0,m}$ is constant, thus (33) follows with $p = p_{0,m}$. The converse is obvious.

For $\beta = 1$ note that $(-1)^{m+1}\widehat{F}^{(m+1)}(s)$ is the Laplace–Stieltjes transform of $\int_0^x y^{m+1}\mathrm{d}F(y)$. Therefore, by Theorem 1, (33) and (34) are equivalent, and $q_{m+1} = p$. □

We spell out this result in the most important special case, when $m = 0$. In this case $f_0(s) = g_0(s) = 1 - \widehat{F}(s)$.

Corollary 1. *Let $\ell \in \mathcal{SV}_\infty$, $\alpha \in [0,1]$, $q_0, p \in \mathcal{P}_r$. The following are equivalent:*

$$1 - \widehat{F}(s) \sim s^\alpha \ell(1/s) q_0(s) \text{ as } s \downarrow 0; \tag{41}$$

$$\begin{cases} \lim_{n\to\infty} \frac{(r^n z)^\alpha}{\ell(r^n z)} \overline{F}(r^n z) = p(z) \text{ for each } z \in C_p, & \alpha \in [0,1), \\ \lim_{n\to\infty} \ell(r^n)^{-1} \int_0^{r^n z} y \mathrm{d}F(y) = p(z) \text{ for each } z \in C_p, & \alpha = 1. \end{cases} \tag{42}$$

If $\alpha > 0$, then (41), (42) are further equivalent to

$$-\widehat{F}'(s) \sim s^{\alpha-1} \ell(1/s) q_1(s) \text{ as } s \downarrow 0, \tag{43}$$

and $q_1 = \mathrm{B}_\alpha^{-1} q_0$.

Moreover, $p = \mathrm{B}_{1-\alpha}^{-1} \mathrm{A}_{1-\alpha}^{-1} q_0$, if $\alpha \in (0,1)$. If $\alpha \in \{0,1\}$, then necessarily $p(x) \equiv p > 0$, $q_0(s) \equiv q_0 > 0$, and $p = q_0$.

Example 2. St. Petersburg distribution. The random variable X has generalized St. Petersburg distribution with parameter $\alpha \in (0,1]$ (and $p = q = 1/2$) if $\mathbf{P}\{X = 2^{n/\alpha}\} = 2^{-n}$, $n = 1, 2, \ldots$. The tail of the distribution function

$$\overline{F}(x) = \mathbf{P}\{X > x\} = \frac{2^{\{\alpha \log_2 x\}}}{x^\alpha}, \quad x \geq 2^{1/\alpha},$$

where $\{x\}$ stands for the fractional part of x. On generalized St. Petersburg distributions we refer to Csörgő [13], Berkes, Györfi, and Kevei [3], and the references therein.

With the notation of Corollary 1, for $\alpha < 1$ we have $r = 2^{1/\alpha}$, $p(z) \equiv 2^{\{\alpha \log_2 z\}}$, and $\ell(x) \equiv 1$, while if $\alpha = 1$ then $r = 2$, $p(z) \equiv 1$, and $\ell(x) = \log_2 x$. In this special case for the Laplace transform

$$\widehat{F}(s) = \sum_{n=1}^\infty e^{-2^{n/\alpha} s} 2^{-n}$$

explicit computation shows that

$$\begin{aligned} 1 - \widehat{F}(s) &\sim s^\alpha \sum_{m=-\infty}^\infty \left(1 - \exp\left[2^{(m - \{\alpha \log_2 s^{-1}\})/\alpha}\right]\right) 2^{-m + \{\alpha \log_2 s^{-1}\}} \\ &=: s^\alpha q_0(s) \end{aligned}$$

as $s \downarrow 0$, whenever $\alpha < 1$, and

$$1 - \widehat{F}(s) \sim s \log_2 s^{-1} \quad \text{as } s \downarrow 0,$$

for $\alpha = 1$. This is exactly the statement of Corollary 1. A somewhat lengthy but straightforward calculation shows that $q_0 = A_{1-\alpha} B_{1-\alpha} p$ for $\alpha < 1$.

3.2 Fixed points of smoothing transforms

Let $T = (T_i)_{i \in \mathbb{N}}$ be a sequence of nonnegative random variables; it can be finite, or infinite, dependent, or independent. A random variable X, or its distribution, is the fixed point of the (homogeneous) smoothing transform corresponding to T, if

$$X \stackrel{\mathcal{D}}{=} \sum_{i \geq 1} X_i T_i, \tag{44}$$

where on the right-hand side $X_1, X_2, \ldots$ are iid copies of X, and they are independent of T.

The theory of smoothing transforms goes back to Mandelbrot [28]. Existence and behavior of the solution of equations of type (44) was investigated by Durrett and Liggett [14], Guivarc'h [19], Liu [27], Biggins and Kyprianou [5], Alsmeyer, Biggins, and Meiners [1], to mention just a few. For applications and references we refer to Section 5.2 in the monograph [11] by Buraczewski, Damek, and Mikosch.

Most of the results on the tail behavior of the solution provide conditions which imply exact power-law tail. We are aware of very few exceptions. Theorem 2.2 in [27] states that in the arithmetic case, under appropriate conditions there is an $\alpha > 0$, such that

$$0 < \liminf_{x\to\infty} x^\alpha \mathbf{P}\{X > x\} \le \limsup_{x\to\infty} x^\alpha \mathbf{P}\{X > x\} < \infty.$$

Guivarc'h [19, p.268] noted without proof that in the arithmetic case under appropriate conditions the tail of X, the solution of (44) behaves as $p(x)x^{-\alpha}$, for some $p \in \mathcal{P}_{r,\alpha}$. The implicit renewal theory for the (nonhomogeneous) smoothing transform was worked out by Jelenković and Olvera-Cravioto [23] both in the arithmetic case and nonarithmetic case.

In order to state the main result in [1] we need some further definition and assumptions. Let $N = \sum_i I(T_i > 0)$ denote the number of positive terms in the right-hand side in (44) and put $m(\theta) = \mathbf{E}\sum_{i=1}^N T_i^\theta$. Assume that

(i) $\mathbf{P}\{T \in \{0,1\}^{\mathbb{N}}\} < 1$;

(ii) $\mathbf{E}N > 1$;

(iii) there exists an $\alpha \in (0,1]$, such that $1 = m(\alpha) < m(\beta)$, for each $\beta \in [0,\alpha)$;

(iv) either $\mathbf{E}\sum_{i\ge1} T_i^\alpha \log T_i \in (-\infty, 0)$ and $\mathbf{E}(\sum_{i\ge1} T_i^\alpha)\log_+ \sum_{i\ge1} T_i^\alpha < \infty$, or there is a $\theta \in [0,\alpha)$, such that $m(\theta) < \infty$;

(v) there exists a nonnegative random variable W, which is not identically 0, such that

$$W \stackrel{\mathcal{D}}{=} \sum_{i\ge1} T_i^\alpha W_i,$$

where on the right-hand side $W_1, W_2, \ldots$ are iid copies of W, they are independent of T, and T has the same distribution as in (44);

(vi) the positive elements of T are concentrated on $r^{\mathbb{Z}}$ for some $r > 1$, and r is the smallest such number.

Under the above assumptions in [1, Corollary 2.3] it was showed that the Laplace transform φ of the solution of the fixed point equation (44) has the form

$$\varphi(t) = \psi(h(t)t^\alpha), \quad t \ge 0, \tag{45}$$

where $\alpha \in (0,1]$, h is a logarithmically r-periodic function such that $h(t)t^\alpha$ is a Bernstein-function, i.e. its derivative is completely monotone, and ψ is a Laplace transform of the random variable W in (v), such that $(1-\psi(t))t^{-1}$ is slowly varying at 0.

The tail behavior of the solutions was not discussed in [1]. Theorem 4, in particular Corollary 1, allows us the determine the tail behavior of such solutions. Indeed, if $\ell \in \mathcal{SV}_\infty$, then $\tilde{\ell}(x) = \ell(x^\alpha h(x)) \in \mathcal{SV}_\infty$. Therefore, from (45)

$$1 - \varphi(t) = t^\alpha \tilde{\ell}(1/t) h(t),$$

which allows us to apply Corollary 1. Noting that $\ell(x^\alpha h(x)) \sim \ell(x^\alpha)$ as $x \to \infty$, we obtain the following.

Corollary 2. *Assume (i)–(vi). If $\alpha < 1$, then for the tail $\overline{F}(x) = \mathbf{P}\{X > x\}$ of the solution of equation (44) we have*

$$\lim_{n\to\infty} \frac{(r^n z)^\alpha}{\ell(r^{\alpha n})} \mathbf{P}\{X > r^n z\} = p(z), \quad z \in C_p,$$

where $p = \mathrm{B}_{1-\alpha}^{-1} \mathrm{A}_{1-\alpha}^{-1} h$. While, if $\alpha = 1$, then $h(t) \equiv h$ is necessarily a constant, and

$$\int_0^x y \mathrm{d}F(y) \sim h\,\ell(x) \quad \textit{as } x \to \infty.$$

3.3 Semistable laws

Logarithmically periodic functions, and regularly log-periodic functions naturally arise in the analysis of semistable distributions. The class of semistable laws, introduced by Paul Lévy, is an important subclass of infinitely divisible laws. The semistable laws are the stable laws, and those infinitely divisible distributions, which has no normal component, and the Lévy measure μ in the Lévy–Khinchin representation satisfies

$$\mu((x,\infty)) = x^{-\alpha} p_+(x), \ \mu((-\infty,x)) = x^{-\alpha} p_-(x), \ x > 0,$$

where $\alpha \in (0,2)$, $r > 1$, and $p_+, p_- \in \mathcal{P}_{r,-\alpha} \cup \{0\}$ (0 is the identically 0 function), such that at least one of them is not identically 0. For properties, characterization, applications and some history of semistable laws we refer to Megyesi [30], Huillet, Porzio, and Ben Alaya [21], and Meerschaert and Scheffler [29], and the references therein. For a more recent account on semistability see Chaudhuri and Pipiras [12]. We note that in the characterization of the domain of geometric partial attraction regularly log-periodic functions play an important role; see Grinevich and Khokhlov [18], and Megyesi [30].

Although there has been large interest in semistable laws in the last 50 years, the tail behavior was determined completely only in 2012 by Watanabe and Yamamuro [35]; for partial results for nonnegative semistable distributions see [21, p.357] with continuous p function, and Shimura and Watanabe [31, Theorem 1.3] with general p. We reprove some of the results in [35], emphasizing that more precise and more general results were shown in [35]. In particular, we restrict ourselves to the nonnegative semistable laws, since the technique developed in this paper works only for one-sided laws.

The Laplace transform of a nonnegative semistable random variable W has the form

$$\mathbf{E} e^{-sW} = \exp\left\{ -as - \int_0^\infty (1 - e^{-sy}) \nu(\mathrm{d}y) \right\}, \tag{46}$$

where $a \geq 0$, and ν is a Lévy measure such that $\overline{\nu}(x) = p(x)x^{-\alpha}$, with $p \in \mathcal{P}_{r,-\alpha}$, $\alpha \in (0,1)$, and $\overline{\nu}(x) = \nu((x,\infty))$, $x > 0$. Partial integration gives

$$\int_0^\infty (1 - e^{-sy})\nu(\mathrm{d}y) = \int_0^\infty e^{-sy} s\overline{\nu}(y)\mathrm{d}y = s\widehat{U}(s),$$

where

$$U(x) = \int_0^x \overline{\nu}(y)\mathrm{d}y = x^{1-\alpha}\mathrm{B}_{1-\alpha}p(x).$$

From Theorem 1 we have

$$\widehat{U}(s) \sim s^{\alpha-1}q(s) \quad \text{as } s \downarrow 0,$$

with $q = \mathrm{A}_{1-\alpha}\mathrm{B}_{1-\alpha}p$. Thus, (46) gives

$$1 - \mathbf{E}e^{-sW} \sim as + \int_0^\infty (1 - e^{-sy})\nu(\mathrm{d}y) \sim s^\alpha q(s) \quad \text{as } s \downarrow 0.$$

Corollary 1 implies

$$\lim_{n\to\infty} (r^n z)^\alpha\, \mathbf{P}\{W > r^n z\} = p(z) \quad \text{for each } z \in C_p,$$

or, which is the same

$$\lim_{n\to\infty} r^{n\alpha}\, \mathbf{P}\{W > r^n z\} = \overline{\nu}(z) \quad \text{for each } z \in C_p.$$

This is the statement in Theorem 1 [35]. However, there the limit above is determined for any $z > 0$.

3.4 Supercritical Galton–Watson process

Consider a supercritical Galton–Watson process $(Z_n)_{n\in\mathbb{N}}$, $Z_0 = 1$, with offspring generating function $f(s) = \mathbf{E}s^{Z_1}$, and offspring mean $\mu = \mathbf{E}Z_1 \in (1,\infty)$. Let $q \in [0,1)$ denote the extinction probability, i.e. the smaller root of $f(s) = s$ in $[0,1]$. Denote f_n the n-fold iterate of f, which is the generating function of Z_n. On general theory of branching processes see Athreya and Ney [2].

Further assume $\mathbf{E}Z_1 \log Z_1 < \infty$, which assures that

$$\frac{Z_n}{\mu^n} \longrightarrow W \quad \text{as } n \to \infty \text{ a.s.}, \tag{47}$$

with $\mathbf{E}W = 1$. The Laplace transform of W, $\varphi(t) = \mathbf{E}e^{-tW}$, $t \geq 0$, satisfies the Poincaré functional equation

$$\varphi(\mu t) = f(\varphi(t)).$$

The latter equation always has a unique (up to scaling) solution, which is a Laplace transform of a distribution. However, the law of W can be determined explicitly only in very few special cases. Therefore, it is important to obtain asymptotic behavior of the tail probabilities. Assume that we are in the Schröder case, that is $\gamma = f'(q) > 0$. Put $\alpha = -\log\gamma/\log\mu$. Harris [20, Theorem 3.3] proved that

$$\varphi(s) - q \sim \frac{K(s)}{s^\alpha} \quad \text{as } s \to \infty, \tag{48}$$

where K is a logarithmically periodic function with period μ. Note that the limit distribution in (47) puts mass q at 0, therefore $\lim_{s\to\infty} \varphi(s) = q$. From a version of Theorem 1, with $n \to -\infty$ in (10) and $s \to \infty$ in (11), it follows for the distribution function $G(x) - q = \mathbf{P}\{W \leq x\}$ that

$$\lim_{n\to\infty} [G(r^{-n}z) - q](r^{-n}z)^{-\alpha} = p(z), \tag{49}$$

with $p = \mathrm{A}_\alpha^{-1} K$.

A much stronger result was shown by Biggins and Bingham [4, Theorem 4], namely

$$G'(x) \sim x^{\alpha-1} V(x) \quad \text{as } \ x \downarrow 0,$$

where V is a continuous, positive, logarithmically periodic function with period μ. For further results on tail asymptotics of W we refer to Bingham [6], Biggins and Bingham [4], and to the more recent papers by Fleischmann and Wachtel [16] and by Wachtel, Denisov, and Korshunov [34].

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